A coloring algorithm for $4K_1$-free line graphs

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Abstract

Let L be a set of graphs. *Free*(L) is the set of graphs that do not contain any graph in L as an induced subgraph. It is known that if L is a set of four-vertex graphs, then the complexity of the coloring problem for *Free*(L) is known with three exceptions: $L = \{\text{claw}, 4K_1\}$, $L = \{\text{claw}, 4K_1, \text{co-diamond}\}$, and $L = \{C_4, 4K_1\}$. In this paper, we study the coloring problem for *Free*(claw, $4K_1$). We solve the coloring problem for a subclass of *Free*(claw, $4K_1$) which contains the class of $4K_1$-free line graphs. Our result implies the chromatic index of a graph with no matching of size four can be computed in polynomial time.
Keywords: Graph coloring, *claw*, $K_5 - e$, *line graph*

1 Introduction

Graph coloring is one of the most important problems in graph theory and computer science. Determining the chromatic number of a graph is a NP-hard problem. However, for some graph families the problem can be solved in polynomial time. Let L be a set of graphs. Define *Free*(L) to be the class of graphs that do not contain any graph in the list L as an induced subgraph. For example, *Free*(P_4) is the class of graphs that do not contain a P_4 as an induced subgraph; and *Free*(P_5, co-P_5) is the class of graphs that do not contain an induced subgraph isomorphic to a P_5 or the complement of a P_5. For a single graph H, in [14], it is proved the coloring problem for *Free*(H) is polynomial time solvable if H is the P_4, or the join of a P_3 and P_1, and NP-complete for any other graph H. This result motivates us to consider the problem of coloring the class *Free*(L) when L is a family of four-vertex graphs. As this paper is being written, we discovered that [15] has considered the same problem. We found

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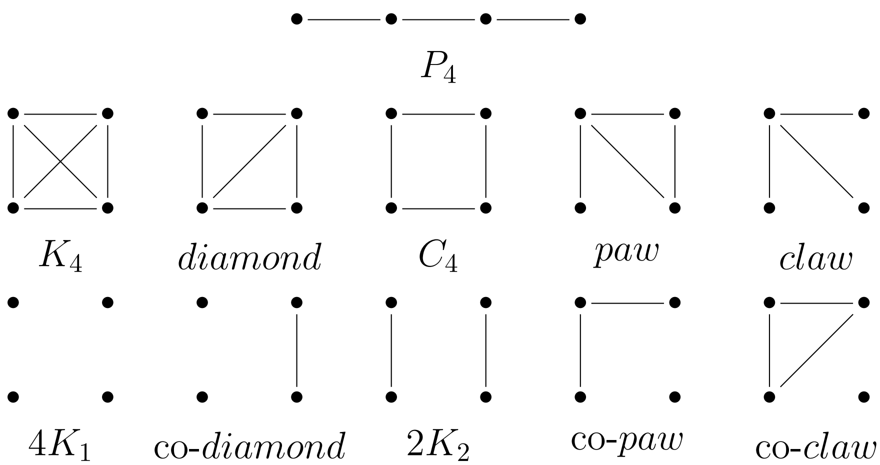

Figure 1: All 4-vertex graphs

some results already discovered in [15], but we also found a new result which we will present in this paper. To explain this result, we will need to discuss the background of the problem. Let VERTEX COLORING be the problem of determining the chromatic number of a graph. For graphs G and H, $G + H$ denotes the disjoint union of G and H. For a graph G and an integer k, kG denotes the disjoint union of k copies of G. Let P_n (respectively, C_n, K_n) denote the chordless path (respectively, chordless cycle, clique) on n vertices.

Recall the following result in [14]:

Theorem 1.1 *For a single graph H, VERTEX COLORING is polynomial time solvable for Free(H) if H is the P_4, or $P_3 + P_1$, and NP-Complete otherwise.*

In [19], the following result is established (see [19] for the definition of clique widths. For the purposes of this paper, we need only know the fact that if the clique width of a graph is bounded then so is that of its complement.)

Theorem 1.2 *VERTEX COLORING is polynomial time solvable for graphs with bounded clique width.*

In [5], the authors study the clique widths of *Free*(F) where F is a family of four-vertex graphs. Figure 1 shows all 11 graphs on four vertices with their names. They show there are exactly seven minimal classes with unbounded clique width. These are:
$\mathcal{X}_1$ = *Free*(claw, C_4, K_4, diamond).
$\mathcal{X}_2$ = *Free*(co-claw, $2K_2$, $4K_1$, co-diamond).
$\mathcal{X}_3$ = *Free*(C_4, co-claw, paw, diamond, K_4).
$\mathcal{X}_4$ = *Free*($2K_2$, claw, co-paw, co-diamond, $4K_1$).

$\mathcal{X}_5 = \mathit{Free}(K_4,\, 2K_2)$.
$\mathcal{X}_6 = \mathit{Free}(C_4,\, 2K_2)$.
$\mathcal{X}_7 = \mathit{Free}(C_4,\, 4K_1)$.

Thus, if F is a set of four-vertex graphs and $F \not\subseteq \mathcal{X}_i$ $(i = 1, 2, \ldots, 7)$, then VERTEX COLORING is polynomial time solvable for *Free*(F).

VERTEX COLORING is NP-Complete for

- $\mathcal{X}_1$ due to a result in [14] which shows the problem is NP-Complete for *Free*(claw, C_4) and for *Free*(claw, K_4, diamond);
- $\mathcal{X}_2$ due to Theorem 6 in [20];
- $\mathcal{X}_3$ due to a remark (Case 1 in Section 4) in [14]: they show the problem is NP-Complete for *Free*(L) if every graph in L contains a cycle .

Thus, VERTEX COLORING is NP-Complete for *Free*(L) whenever $L \subseteq \mathcal{X}_i$, for $i = 1, 2, 3$. Therefore, we only need to examine the problem for classes $\mathcal{X}_4$, $\mathcal{X}_5$, $\mathcal{X}_6$, $\mathcal{X}_7$ and their super classes defined by forbidding induced subgraphs with four vertices. In [15], a polynomial time algorithm is given for VERTEX COLORING for the class $\mathcal{X}_5$. The graphs in $\mathcal{X}_6$ have a simple structure [4] that implies an easy polynomial time algorithm for the coloring problem. Furthermore, in [11] a polynomial time algorithm for VERTEX COLORING for the larger class *Free*(P_5, co-P_5) is given. The complexity of VERTEX COLORING for the class $\mathcal{X}_7$ is unknown. In [15], the authors conjecture that the coloring problem can be solved in polynomial time for $\mathcal{X}_7$.

We are interested in the class $\mathcal{X}_4$. Let H be a subset of $\{2K_2$, claw, co-paw, co-diamond, $4K_1\}$. We examine the complexity of VERTEX COLORING for *Free*(H). We may assume H does not contain a co-paw, for otherwise the problem is polynomial time solvable, by Theorem 1.1. We may assume H contains the claw, for otherwise $H \subset \mathcal{X}_2$, and so VERTEX COLORING is NP-Complete. In [15], a polynomial time algorithm is given for VERTEX COLORING for class *Free*(claw, $2K_2$). Thus we have $H \subseteq \{$claw, co-diamond, $4K_1\}$. In [15], it is proved VERTEX COLORING for *Free*(claw, co-diamond) is polynomially equivalent to the same problem for the class *Free*(claw, co-diamond, $4K_1$). Thus, if VERTEX COLORING is polynomial time solvable for *Free*(claw, $4K_1$), then so is the same problem for *Free*(claw, co-diamond). These are two challenging problems. We believe VERTEX COLORING can be solved in polynomial time for *Free*(claw, $4K_1$). The purpose of this paper is to solve the problem for a subclass of *Free*(claw, $4K_1$): the class of $4K_1$-free line graphs.

Given a graph G, the line graph $L(G)$ of G is defined to be the graph whose vertices are the edges of G, and two vertices of $L(G)$ are adjacent if their corresponding edges in G are incident. Line graphs cannot contain a claw. In [1], a characterization of line graphs by forbidden induced subgraphs is given: A graph is a line graph (of some other graph) if and only if it does not contain a graph in Figure 2 as an induced

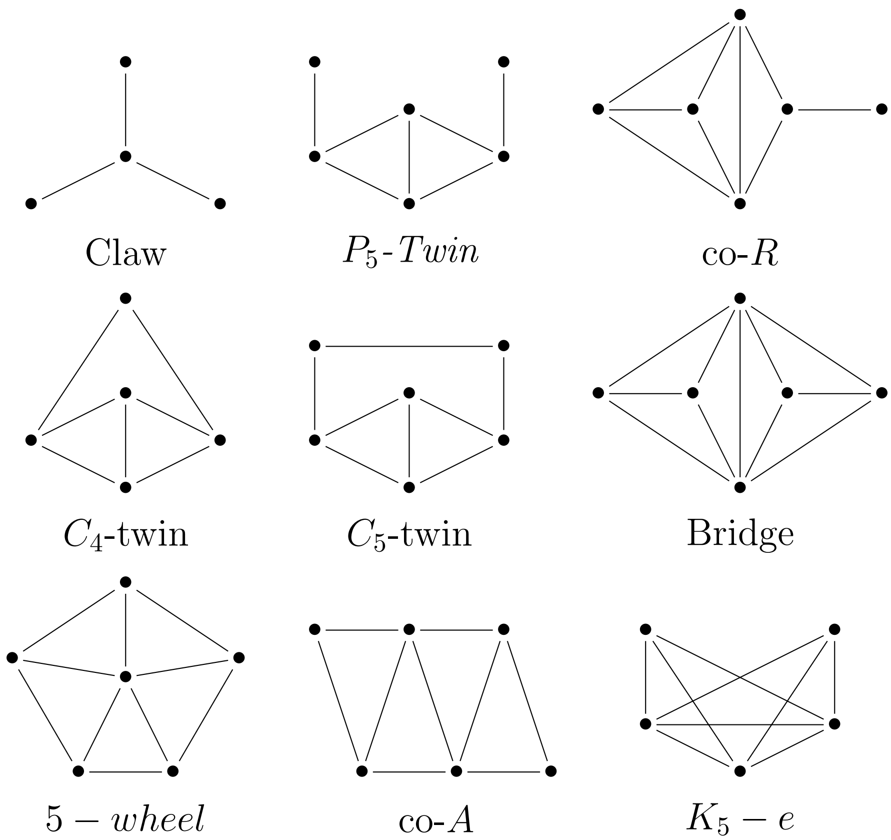

Figure 2: All minimal forbidden induced subgraphs for line graphs

subgraph. The purpose of this paper is to prove the following theorem (see Figure 2 for the names of the graphs mentioned in our theorem.)

Theorem 1.3 *VERTEX COLORING is polynomial time solvable for Free(claw, $4K_1$, 5-wheel, C_5-twin, P_5-twin, $K_5 - e$).*

An *edge coloring* of a graph is an assignment of colors to its edges such that every edge receives one color and two edges receive different colors if they are incident. The *chromatic index* of a graph is the smallest number of colors needed to color its edges. Let $\Delta(G)$ be the maximum degree of G. Then the chromatic index of a graph G is at least $\Delta(G)$. A classic theorem of Vizing [22] states that the chromatic index of a graph G is at most $\Delta(G) + 1$. However, computing the chromatic index of a graph is NP-hard [12]. Let G be a graph and $L(G)$ be its line graph. Then the chromatic index of G is equal to the chromatic number of $L(G)$. A matching of G is a set of edges such that no two edges in it are incident; this matching of G corresponds to a stable set of the line graph $L(G)$. It follows that our Theorem 1.3 implies the

chromatic index can be computed in polynomial time for graphs without matching of size four.

Corollary 1.4 *There is a polynomial-time algorithm to compute the chromatic index of a graph without a matching of size four.*

In Section 2, we discuss the background results needed to prove our main theorem. In Section 3, we establish structural properties of the graphs in *Free*(claw, $4K_1$, 5-wheel, C_5-twin, P_5-twin, $K_5 - e$). In Section 4, we prove Theorem 1.3. And finally, in Section 5, we discuss open problems related to our work.

2 Definitions and background

In this section, we discuss the background of our problem. Let G be a graph. Then *co-G* denotes the complement of G. A *clique cutset* of G is a set of vertices S where $G[S]$, the subgraph of G induced by S, is a clique whose removal disconnects G. An *atom* is a connected graph containing no clique cutset. Let $\chi(G)$ denote the chromatic number of a graph G. Let $\omega(G)$ denote the number of vertices in a largest clique of G. Let $\alpha(G)$ denotes the number of vertices in a largest stable set of G. A *hole* is the graph C_k with $k \geq 5$. For a hole H, a vertex x outside H is a k-vertex (for H) if x has exactly k neighbours in H. An *anti-hole* is the complement of a hole. A hole is *odd* if it has an odd number of vertices. A graph is *Berge* if it contains no odd holes and no odd anti-holes. For sets X, Y of vertices, we write X 0 Y to mean there is no edge between any vertex in X and any vertex in Y, and X 1 Y to mean there are all edges between X and Y. We let $|G|$ denotes the number of vertices of G. By *claw(a,b,c,d)* we denote the claw with vertices a, b, c, d and edges ab, ac, ad. We say that a stable set is *good* if it meets every largest clique of the graph. A set is *big* if it has at least three elements. For a vertex x of G, $N(x)$ denote the set of neighbors of x, and $d(x)$ is the *degree* of x, that is, $d(x) = |N(x)|$.

Consider the following procedure to decompose a (connected) graph G. If G has a clique cutset C, then G can be decomposed into subgraphs $G_1 = G[V_1]$ and $G_2 = G[V_2]$ where $V = V_1 \cup V_2$ and $C = V_1 \cap V_2$ (recall that $G[X]$ denotes the subgraph of G induced by X for a subset X of vertices of $V(G)$). Given optimal colourings of G_1, G_2, we can obtain an optimal colouring of G by identifying the colouring of C in G_1 with that of C in G_2. In particular, we have $\chi(G) = \max(\chi(G_1), \chi(G_2))$. If G_i ($i \in \{1, 2\}$) has a clique cutset, then we can recursively decompose G_i in the same way. This decomposition can be represented by a binary tree $T(G)$ whose root is G and the two children of G are G_1 and G_2, which are in turn the roots of subtrees representing the decompositions of G_1 and G_2. Each leaf of $T(G)$ corresponds to an atom of G. Algorithmic aspects of the clique cutset decomposition are studied in [21] and [23]. In particular, the decomposition tree $T(G)$ can be constructed in

$O(nm)$ time such that the total number atoms is at most $n-1$ [21] (Here, as usual, n and, respectively, m, denote the number of vertices, respectively, edges, of the input graph). Thus, to color a graph G in polynomial time, we only need to color its atoms in polynomial time.

Our result relies on known theorems on perfect claw-free graphs, and we discuss these results now. A graph G is *perfect* if for each induced subgraph H of G, we have $\chi(H) = \omega(H)$. In [17], it is proved claw-free Berge graphs are perfect. Claw-free perfect graphs can be recognized in polynomial time [8], and they can be optimally colored in polynomial time [13]. We note [7] proves that a graph is perfect if and only if it is Berge, solving a long standing conjecture of [2]. Perfect graphs can be recognized in polynomial time [6], and they can be optimally colored in polynomial time [10]. In [8], the following result, crucial to our algorithm, is established.

Lemma 2.1 *Let G be a connected claw-free graph with $\alpha(G) \geq 3$. If G contains an odd anti-hole then G contains a C_5.*

Finally, we note the well known observation that VERTEX COLORING is polynomial time solvable for graphs G with $\alpha(G) = 2$; it is sufficient to find a maximum matching in the complement of G.

3 Structural properties

In this section, we establish preliminary results needed to prove Theorem 1.3. For the Claims in this section, we will assume $G = (V, E)$ is a connected graph in *Free*(claw, $K_5 - e$, 5-wheel, C_5-twin, P_5-twin, $4K_1$). We begin with the following easy Claim.

Claim 3.1 *If $\alpha(G) \leq 4$, then G contains no C_ℓ, $\ell \geq 8$.*

Proof. If G contains C_ℓ, $\ell \geq 8$, then G contains a $4K_1$. □

Let C be a hole with vertices $1, 2, \ldots, k$ (in the cyclic order) with $k \geq 5$ (with respect to C, the vertices i will be taken modulo k). Define Y_i be the set of vertices with neighbors $i, i+1, i+2, i+3$. Let Z_i be the set of vertices with neighbors $i, i+1, i+3, i+4$.

Claim 3.2 *If G contains a C_7 , then $|G| \leq 21$.*

Proof. Suppose G contains a C_7. Then G has no k-vertex for $k \in \{0, 1, 2\}$ since G is $4K_1$-free, has no k-vertex for $k \in \{5, 6, 7\}$ since G is claw-free, and G has no 3-vertex, or else G contains a claw or a P_5-twin. Thus, only 4-vertices may exist and they are one of the two types Y_is, Z_is defined above. Let y_1 and y_2 be two vertices in

Y_i, $y_1 \neq y_2$. If $y_1y_2 \notin E$ then there is a $claw(i+3, i+4, y_1, y_2)$ and if $y_1y_2 \in E$ then there is a $K_5 - e$ induced by $\{i, i+1, i+2, y_1, y_2\}$; so we have $|Y_i| \leq 1$. Let z_1 and z_2 be two vertices in Z_i, $z_1 \neq z_2$. If $z_1z_2 \notin E$ then there is a $claw(i+4, i+5, z_1, z_2)$ and if $z_1z_2 \in E$ then there is a C_5-twin induced by $\{i, i+6, i+5, i+4, z_1, z_2\}$. It follows that G contains at most 21 vertices. □

Claim 3.3 *If G contains a C_5 then G has no k-vertex for $k \in \{1, 3, 5\}$.*

Proof. Suppose G contains a C_5. Then G has no k-vertex for $k = 1$ for otherwise G contains a claw, or for $k = 3$ for otherwise G contains a claw or a C_5-twin, or for $k = 5$ for otherwise G contains a 5-wheel. □

For the all the Claims below, we will assume G contains a C_5. Let the 0-vertex set be denoted by R, let X_i be the set of 2-vertices with neighbours $i, i+1$ and let Y_i be the set of 4-vertices with neighbors $i, i+1, i+2, i+3$. Let X denote the set of all 2-vertices and Y denote the set of all 4-vertices. Every vertex of $G - C_5$ belongs to $X \cup Y \cup R$.

Claim 3.4 *We have $|Y_i| \leq 1$ for all i.*

Proof. Let y_1 and y_2 be two vertices in Y_i, $y_1 \neq y_2$. If $y_1y_2 \notin E$ then there is a $claw(i, i-1, y_1, y_2)$. If $y_1y_2 \in E$ then there is a $K_5 - e$ induced by $\{i, i+1, i+2, y_1, y_2\}$. □

From Claim 3.4, we have $|Y| \leq 5$.

Claim 3.5 *We have Y_i 0 Y_{i+1} for all i.*

Proof. Let y_1 be the vertex from Y_i and y_2 be the vertex from Y_{i+1}. If $y_1y_2 \in E$ then there is a $K_5 - e$ induced by $\{i+1, i+2, i+3, y_1, y_2\}$. □

For the following two claims, we will let x_i (respectively, y_i) denote an arbitrary vertex in X_i (respectively, Y_i) for all i.

Claim 3.6 X_i 1 $Y_i \cup Y_{i+3}$.

Proof. If $x_iy_i \notin E$, then there is a claw$(i, i+4, x_i, y_i)$. If $x_iy_{i+3} \notin E$, then there is a claw$(i+1, i+2, x_i, y_{i+3})$. □

Claim 3.7 *If $X_i \neq \emptyset$, and both y_i and y_{i+3} exist, then $y_iy_{i+3} \in E$.*

Proof. If $y_i y_{i+3} \notin E$, then by Claim 3.6, the set $\{x_i, i, i+1, y_i, y_{i+3}\}$ contains a $K_5 - e$. $\square$

Claim 3.8 X_i (0) $Y_{i+1} \cup Y_{i+2} \cup Y_{i+4}$.

Proof. If $x_i y_{i+1} \in E$, then there is a claw$(y_{i+1}, x_i, i+2, i+4)$. If $x_i y_{i+4} \in E$, then there is a claw$(y_{i+4}, i+4, i+2, x_i)$. If $x_i y_{i+2} \in E$, then there is a claw$(y_{i+2}, i+2, i+4, x_i)$. $\square$

Claim 3.9 *We have* R (0) Y.

Proof. If some $y \in Y$ is adjacent to some $r \in R$, then there is a claw induced by y, r and some two neighbours a, b of y in the C_5 with $ab \notin E$. $\square$

Claim 3.10 *If* $R \neq \emptyset$, *then* $X \neq \emptyset$. $\square$

Proof. Since G is connected by the assumption of this section, there is a path connecting some vertex in R to some vertex in the C_5. By Claim 3.9, this path must contain some vertex in X. $\square$

Claim 3.11 *Each* X_i *forms a clique for all* i.

Proof. Let v_1 and v_2 be two vertices in X_i, $v_1 \neq v_2$. If $v_1 v_2 \notin E$ then there is a *claw*$(i, i-1, v_1, v_2)$. So $v_1 v_2 \in E$ and X_i forms a clique. $\square$

Claim 3.12 *The set* R *induces a clique.*

Proof. Let r_1 and r_2 be vertices in R. If $r_1 r_2 \notin E$ then the set $\{0, 2, r_1, r_2\}$ induces a $4K_1$. $\square$

Claim 3.13 R (1) X_i *for all* i.

Proof. Let x be a vertex in X_i and r be a vertex in R. If $rx \notin E$ then there is a $4K_1$ induced by $\{r, x, i-1, i+2\}$. $\square$

Claim 3.14 *If* $R \neq \emptyset$ *then* $|X_i| \leq 2$ *for all* i.

Proof. Suppose $|X_i| \geq 3$ and $R \neq \emptyset$. Let $x_1,\ x_2,$ and x_3 be three distinct vertices from X_i. By Claim 3.11, the vertices $x_1,\ x_2,$ and x_3 form a clique. By Claim 3.13 there is a $K_5 - e$ induced by $\{r, x_1, x_2, x_3, i\}$ for a vertex $r \in R$. □

Claim 3.15 *A vertex in X_i cannot have two neighbors in X_j for any two distinct i and j.*

Proof. Suppose some vertex $x_i \in X_i$ is adjacent to two vertices $a, b \in X_j$. We may assume $j = i+1$, or $j = i+2$. Now, there is a P_5-twin induced by $\{i, x_i, a, b, j+1, j+2\}$ if $j = i+1$, or $\{j-1, x_i, a, b, j+1, j+2\}$ if $j = i+2$. □

Claim 3.16 *If $X \neq \emptyset$ then $|R| \leq 2$ or X is a clique cutset of G.*

Proof. Suppose $X \neq \emptyset$ and $|R| \geq 3$. By Claim 3.13, R 1 X. By Claim 3.9, X is a cutset separating R from the C_5. We may assume X contains two non-adjacent vertices v_1, v_2, for otherwise X is a clique cutset and we are done. But now, by Claim 3.12, any three vertices in R together with v_1, v_2 induce a $K_5 - e$ in G. □

Claim 3.17 *If $R \neq \emptyset$ then $|G| \leq 22$ or G contains a clique cutset.*

Proof. If $|R| \geq 3$ then by Claims 3.16 and 3.10, X is a clique cutset of G. So we have $|R| \leq 2$. By Claim 3.14, we have $|X_i| \leq 2$ for $i \in 0, 1, 2, 3, 4$. By Claim 3.4, we have $|Y| \leq 5$. Then $|G| = |R| + |X| + |Y| + |C_5| \leq 2 + 10 + 5 + 5 = 22$. □

Claim 3.18 *A vertex in X_i cannot have two non-adjacent neighbors in $X - X_i$.*

Proof. Suppose some vertex $x_i \in X_i$ have non-adjacent neighbors v_1 and v_2 in $X - X_i$. By Claim 3.11, we have $v_1 \in X_j,\ v_2 \in X_k,\ j \neq k$. If $j = i-1$ and $k = i+1$ then there is a P_5-twin induced by $\{v_2, x_i, v_1, i, j, j-1\}$. Now, we may assume $k \notin \{i-1, i+1\}$, it follows there is a claw$(x_i, v_2, v_1, i+1)$. □

Claim 3.19 *Suppose X_i, X_{i+1}, X_{i+2} are each non-empty for some i. If $|X_j| \geq 3$ then $|X_k| = |X_\ell| = 1$, for $\{j, k, \ell\} = \{i, i+1, i+2\}$.*

Proof. Suppose $|X_j| \geq 3$. Suppose some vertex $x_k \in X_k$ is non-adjacent to some vertex $x_\ell \in X_\ell$. By Claim 3.15, there is a vertex $x_j \in X_j$ that is non-adjacent to both x_k and x_ℓ. But now $\{i+4, x_j, x_k, x_\ell\}$ induces a $4K_1$. Thus, we have X_k 1 X_ℓ. It follows from Claim 3.15 that $|X_k| = |X_\ell| = 1$. □

Claim 3.20 *Suppose $R = Y = X_{i+2} = X_{i+4} = \emptyset$ for some i. Then $\chi(G) = \omega(G)$, and an optimal colouring of G can be found in polynomial time.*

Proof. By induction on the number of vertices. Let X_k be the set with largest cardinality among the three sets X_i, X_{i+1}, X_{i+3}. If $|X_k| = 1$, then $\omega(G) = 3, |V| \leq 8$, and a 3-coloring of G can be trivially found. Similarly, if $|X_k| = 2$, then $\omega(G) = 4$, and a 4-coloring of G can be found. Now, we may assume $|X_k| \geq 3$. From Claims 3.15 and 3.18, there is a stable set S containing a vertex in X_k and some vertices in $X_j \cup X_\ell$ with $k \notin \{j, \ell\}$ that is good, ie., meets all maximum cliques of G. Now, we can recursively color $G - S$ with $\omega(G) - 1$ colors and then give S a new color. □

In proving the main theorem of this paper, we will reduce the problem to list coloring a restricted class of graphs. We will now define some new notions. Given a graph G and a list of colors $L(v)$ for each vertex of v, an L-coloring of G is a (proper) coloring such that each vertex is assigned a color from its list.

Lemma 3.21 *Let $G = (V, E)$ be a graph whose vertices can be partitioned into three cliques Q_1, Q_2, Q_3 such that*

(a) *each vertex in Q_i is adjacent to at most one vertex in Q_j for all i, j with $i \neq j$.*

(b) *if a vertex in Q_i is adjacent to vertices $b \in Q_j, c \in Q_k$, then $bc \in E$ for distinct i, j, k.*

(c) *for each Q_i, there is a list L_i of colors such that*

- **(i)** *all vertices $v \in Q_i$ have the same list $L(v) = L_i$,*
- **(ii)** $|L_i| \geq |Q_i|$,
- **(iii)** *each L_i contains a color d_i such that the three colors d_1, d_2, d_3 are all distinct.*
- **(iv)** *no color appears in all three L_i.*

Then G admits an L-coloring where $L = L_1 \cup L_2 \cup L_3$.

Proof. Let $|L|$ denotes the number of colors in L. We prove the Lemma by induction on $|L|$. Below, i, j, k are distinct indices taken from $\{1, 2, 3\}$.

If some clique Q_i is empty, then it is easy to see the Lemma holds. If $|Q_1| = |Q_2| = |Q_3| = 1$, then color the only vertex in Q_i with color d_i, and we are done. Thus, some Q_i has at least two vertices.

Suppose some Q_i has exactly one vertex v, and $d_i \notin L_j \cup L_k$. In this case, color v with color d_i, remove v from G. By induction, $G - v$ admits an L-coloring, and we are done.

Suppose Q_i has exactly one vertex, $d_i \in L_j$, and Q_j has at least two vertices. Let v be the only vertex in Q_i. Take a vertex $b \in Q_j$ which is not adjacent to v; color v and b with color d_i, remove d_i from L_j (note that d_j remains in L_j). By induction $G - \{v, b\}$ admits an L-coloring, and we are done.

Now, suppose Q_i has exactly one vertex v. From the above, we may assume $d_i \in Q_j$, and $|Q_j| = 1$. It follows that $|Q_k| \geq 2$ and thus, $d_i \notin L_k$. Note that we have $\{d_i, d_j\} \subseteq L_j$. Color v with color d_i, remove color d_i from L_j. By induction $G - v$ admits an L-coloring, and we are done.

We may now assume each Q_i has at least two vertices. Suppose there is a color $c \notin \{d_1, d_2, d_3\}$. If c appears in only one Q_i, then assign to a vertex $v \in Q_i$ the color c, remove c from L_i, and by induction $G - v$ admits an L-coloring and we are done. Now, assume $c \in Q_i$ and $c \in Q_j$. Since each of Q_i and Q_j has at least two vertices, there are non adjacent vertices (by condition (a)) $a \in Q_i$, $b \in Q_j$. Assign to a and b color c, remove color c from L_i and L_j. By induction, $G - \{a, b\}$ admits an L-coloring, and we are done.

So, we may assume every color belongs to $\{d_1, d_2, d_3\}$. It follows that $2 \leq |Q_i| \leq 3$ for any Q_i. By condition (iv), we have $|Q_i| < 3$ for any i. It follows that $|Q_1| = |Q_2| = |Q_3| = 2$. It is now straightforward to verify that G admits an L-coloring. □

Lemma 3.22 *Let G be a graph in Free(claw, $4K_1$, 5-wheel, $K_5 - e$, P_5-twin, C_5-twin) with a C_5. Suppose $R = X_{i+2} = X_{i+4} = \emptyset$ for some i. If $\omega(G) \geq 5$ then $\chi(G) = \omega(G)$ and an optimal coloring of G can be found in polynomial time*

Proof. Suppose $\omega(G) = 5$. For simplicity, we may assume $i = 1$, i.e., the sets X_3 and X_5 are empty. Each of the sets X_i has at most three vertices for $i = 1, 2, 4$ (since $X_i \cup \{i, i+1\}$ induces a clique by Claim 3.11). For each vertex i in the C_5, color vertex i with color $c(i) = i$. Color the vertex y_i (if it exists) with color $c(y_i) = i - 1$ with the subscript i taken modulo 5. Now, we need to extend this coloring to the set X. For X_i, we will construct a list L_i of feasible colors which are compatible with the already colored vertices in $C_5 \cup Y$. Each vertex in X_i will have the list L_i. Consider a non-empty set X_i ($i \in \{1, 2, 4\}$). If $|X_i| = 1$, then set $L_i = \{c(i+3)\}$ (which is the color of vertex $i+3$). If $|X_i| = 2$, then the vertex y_i or the vertex y_{i+3} (or both) does not exist; for otherwise, by Claim 3.7, the induced graph $G[X_i \cup \{i, i+1, y_i, y_{i+3}\}]$ contains a K_6, contradicting the assumption that $\omega(G) = 5$. Let j be the subscript such that y_j ($j \in \{i, i+3\}$) does not exist. Let $L_i = \{c(i+3), c(y_j)\}$. If $|X_i| = 3$, then both vertices y_i, y_{i+3} do not exist for the same reason above. Let $L_i = \{c(i+2), c(i+3), c(i+4)\}$. A coloring of the vertices of X using colors from the lists L_i does not conflict with the already assigned coloring of $C_5 \cup Y$. By Lemma 3.21, X admits an L-coloring. So we have $\chi(G) = 5$.

Now, we may assume $\omega(G) \geq 6$. We will find a good stable set (recall the definition of good stable sets in Section 2). Note that if a maximum clique C of G contains some

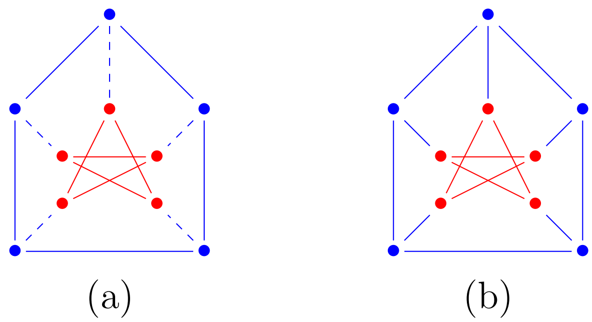

Figure 3: The Petersen graph (b) and its complement (a).

vertex of X_i, then C contains all vertices of X_i. Also, since $\omega(G) \geq 6$, a maximum clique of G does not contain a set X_i with $|X_i| = 1$. Now, since $\omega(G) \geq 6$, some set X_i must have at least three vertices. By Claims 3.15 and 3.18, there is a stable set containing a vertex in every X_j ($j \in \{1, 2, 4\}$) with $|X_j| \geq 2$. Such a set S is the desired stable set. Now, we have $\omega(G - S) = \omega(G) - 1$. We recursively color $G - S$ with $\omega(G) - 1$ colors, and then give vertices in S a new color, and we are done. $\square$

Note that the statement of the Lemma 3.22 is false for $\omega(G) = 4$. There are graphs G in *Free*(claw, $4K_1$, 5-wheel, $K_5 - e$, P_5-twin, C_5-twin) with $\omega(G) = 4$ and $\chi(G) = 5$. The graph in Figure 3 (a) is such a graph. This is the graph with a C_5 (indicated by the outer C_5) and all five y_i vertices (indicated by the inner C_5), there are all edges between the outer and inner C_5s except for the non-edges denoted by the dotted lines. It is very interesting to note that this graph is the complement of the Petersen graph.

4 Coloring algorithm

In the section, we prove Theorem 1.3. Let G be a graph satisfying the hypothesis of the theorem. From the discussion in Section 2, we may assume G is connected, is an atom, ie., G contains no clique cutset, and has $\alpha(G) \geq 3$. Furthermore, if G is perfect, then we are done by [13] or [10]. If G contains an odd anti-hole, then by Lemma 2.1, G contains a C_5. So, we may assume G contains an odd hole H. Since G is $4K_1$-free, H has 5 or 7 vertices. If H has 7 vertices, then by Claim 3.2, $|G|$ is a constant and we are done. So, H is a C_5. From this C_5, define the sets X, Y, R as before, and we may rely on the Claims in Section 3. We may assume $|G|$ is not a constant. By Claim 3.17, we have $R = \emptyset$.

We may assume $\omega(G) \geq 14$; for otherwise, Ramsey's theorem [18] shows that $|G|$ is a constant (both $\omega(G)$ and $\alpha(G)$ are constants, so $|G|$ is a constant). If there is

a vertex v with degree $d(v) \leq 13$, then we recursively color $G - v$ optimally, and then give v a color not appearing in $N(v)$; such a color exists because $\chi(G - v) \geq \omega(G - v) = \omega(G) \geq 14$. So, we may assume every vertex of G has degree at least 14.

Suppose some non-empty X_i has $|X_i| \leq 2$. Then for any vertex $x_i \in X_i$, we have $d(x_i) = |C_5 \cap N(x_i)| + |Y \cap N(x_i)| + |N(x_i) \cap X| \leq 2 + 5 + 5 = 12$ by Claims 3.15 and 3.4. Thus, if X_i is non-empty then it is big (has at least three vertices.)

For a vertex i in the C_5, we may assume X_i or X_{i-1} is big; for otherwise, $d(i) = |C_5 \cap N(i)| + |X_i| + |X_{i+1}| + |Y \cap N(i)| \leq 2 + 2 + 2 + 5 = 11$ by Claim 3.4. Thus, at least three sets X_i's must be big. It follows from Claim 3.19 that precisely three sets X_i are big, and they are not consecutive. By Lemma 3.22, we can color G with $\omega(G)$ colors in polynomial time. $\square$

5 Conclusions and open problems

Let L be a family of four-vertex graphs. As mentioned in Section 1, the complexity of VERTEX COLORING is known for the class *Free*(L) with three exceptions: L = {claw, $4K_1$}, L = {claw, $4K_1$, co-diamond }, and L = {C_4, $4K_1$}. We believe each of the three problems is polynomial time solvable. In this paper, we studied the problem for *Free*(claw, $4K_1$). We solved the coloring problem for a subclass, the class of $4K_1$-free line graphs. Our result implies the chromatic index of a graph with no matching of size 4 can be computed in polynomial time. This is an interesting result in its own right. We conclude our paper with the two open problems below.

Problem 5.1 *For each fixed k, is there a polynomial time algorithm to compute the chromatic index of a graph without a matching of size k?*

Problem 5.2 *For each fixed k, is there a polynomial time algorithm to solve VERTEX COLORING for the class Free(claw, kK_1)?*

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